ETMAG

LECTURE 9

- Continuous functions cont'd
- Derivative of a function at a point

Theorem. (Intermediate Value Theorem, Darboux Theorem)
If a function f is continuous on a closed interval $[a, b]$ then f takes on every value between $f(a)$ and $f(b)$.

To be more precise: if f is continuous on $[a, b]$ then for every $y_0 \in [f(a), f(b)]$ (or $y_0 \in [f(b), f(a)]$, if $f(b) \leq f(a)$) there exists $x_0 \in [a, b]$ such that $f(x_0) = y_0$.

The proof is beyond the scope of this course.

Corollary.
If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign then there exists at least one $x \in [a, b]$ such that $f(x) = 0$.

Corollary. (of corollary)
Every polynomial of an odd degree has at least one (real) root.

Theorem. (Extreme Value Theorem)
If f is continuous on a closed interval $[a,b]$ then it takes on its largest and its smallest values on the interval.

More precisely: there exist $x_{max}, x_{min} \in [a;b]$ such $f(x_{max}) = \sup(f([a,b]))$ and $f(x_{min}) = \inf(f([a,b]))$.

The proof is beyond the scope of this course.

The theorem may be re-phrased as:

If f is continuous on a closed interval $[a;b]$ *then* $f([a;b])$ *is also a closed interval.*

An illustration of the principle.

$f(x) = \frac{1}{x}$ is continuous on (0,1) but not on [0,1]. The set of values is unbounded from above, so $f(x)$ does not assume its largest value (because there is no largest value). On the other hand, it is bounded from below by 1 and the set of lower bounds has the largest element, namely 1. But 1 is not a value for this function at any point in the open interval (0,1). It is, of course, the value of $f(x)$ for $x = 1$ which, however, does not belong to (0;1) (but does to [0;1]). We can also write $f\big((0,1)\big) = (1, +\infty)$

Comprehension.

What is tan([0;π])?

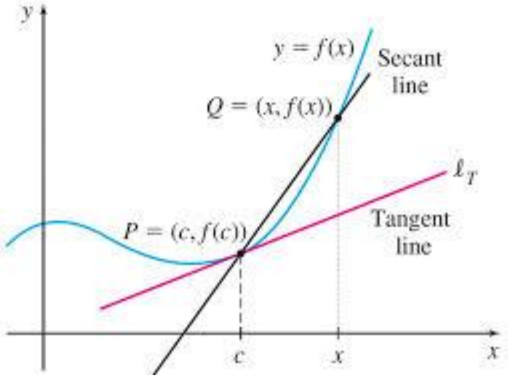
Another illustration of the principle.
$f(x) = x^2$ is continuous on $(-1,2)$ but it has no largest value in $(-1,2)$. Of course $\sup\left(f\left((-1,2)\right)\right) = \sup\left((0,4)\right) = 4$ and $4 = f(2)$ but $2 \notin (-1,2)$.

A continues function f on a <u>closed</u> interval I is guaranteed to assume its largest and smallest values on I at some points of I.

If I is not a closed interval, f <u>may</u> assume its extreme values in I, but it's not guaranteed to.

In the above example, x^2 assumes its smallest value on $(-1,2)$, which is 0, at $x = 0$.

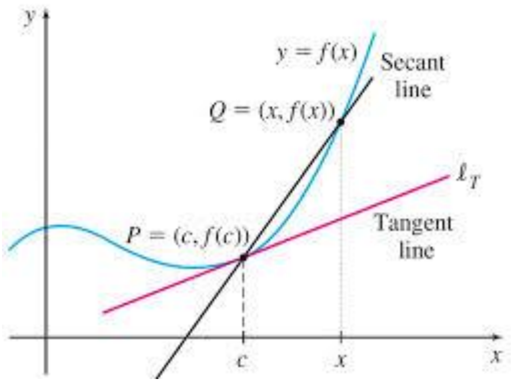
How to find those points in an interval where a function assumes its extreme values (*locally extreme values*) is, however, a completely different story.



(Not-very-formal) **Definition.**

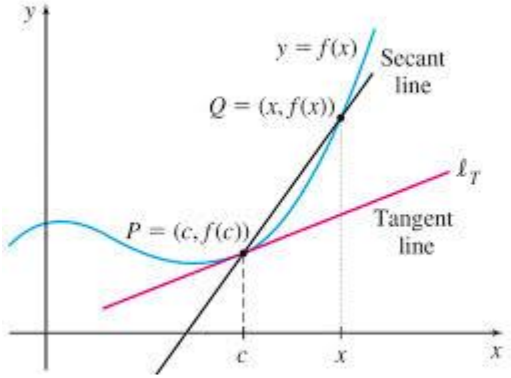
The *tangent line* (or just *tangent*) to the graph of a function $f(x)$ at a point c is the straight line $y = g(x) = px + q$ whose slope p is the *local rate of growth* of the function $f(x)$ at c and whose Y-intercept q is such that $g(c) = f(c)$.

The second part means that the point $\big(c, f(c)\big)$ belongs to both the graph of $y = f(x)$ and the line $y = px + q$.



Problem.
Given a function $f(x)$ and a point c from its domain how do we compute the slope p of the tangent to $f(x)$ at c (if a tangent exists)?

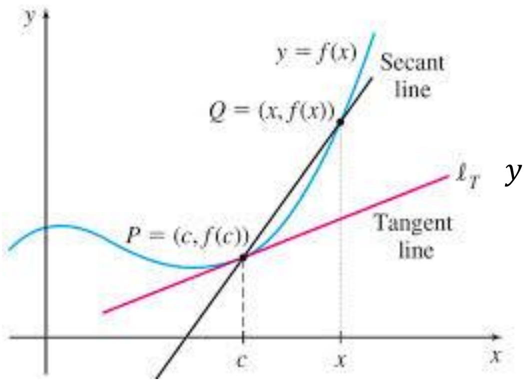
Consider the secant to the curve passing through points $P = \left(c, f(c)\right)$ and $Q = \left(x, f(x)\right)$. It *stands to reason* (tricky phrase!) that the closer x gets to c, the better the slope of the line PQ approximates the slope of the tangent.



Sounds familiar?
Yes, it smells of a limit from miles away!

Definition.
Let $f(x)$ be a function and let c be a point in its domain. If there exists a number $L = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ then f is said to be *differentiable* at c and L is called the *derivative of* $f(x)$ *at* c and is denoted by $f'(c)$. Otherwise $f(x)$ is *non-differentiable at* c.

$f'(c)$ is the slope of the tangent to the graph of $f(x)$ at c.



$y = f'(c)(x - c) + f(c)$

Recall that the *tangent* to the graph of $f(x)$ at a point c is the line $y = g(x) = f'(c)x + q$ such that $p = f'(c)$ where q is such that $g(c) = f(c)$. Hence, $f(c) = f'(c)c + q$, which means $q = f(c) - f'(c)c$. Finally

$y = f'(c)x + f(c) - f'(c)c = f'(c)(x - c) + f(c)$.

Fact.

If $f'(c)$ exists, then the equation of the tangent to the graph of f at c is given by

$$y = f'(c)(x - c) + f(c)$$

We can define the derivative of a function in a slightly different way:

Definition.

For every x such that the limit $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exists, the limit is called *the derivative of* f *at* x and is denoted by $f'(x)$.

Now, f' is a function which assigns to x the derivative of f at x.

Notice that $f'(x)$ may not exist at some points belonging to the domain of f.

Note. The alternate symbol of the derivative function f' is $\frac{df}{dx}$.

Examples. Find the derivative of $f(x)$

1. $f(x) = |x|$.

For every $x > 0$, $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{x+h-x}{h} = 1$.

For every $x < 0$, $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{-x-h+x}{h} = -1$.

At $x = 0$, $\lim\limits_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0^+} \frac{h-0}{h} = 1$ while

$\lim\limits_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0^+} \frac{-h-0}{h} = -1$.

Hence, $|x|$ is not differentiable at 0.

This example leads to the concept of *one-sided derivatives* f'_+ and f'_- and, consequently, to the fact that $f'(c)$ exits iff f'_+ and f'_- exist at c and $f'_+(c) = f'_-(c)$.

Examples (cont'd)

2. $f(x) = x$.

   For every x, $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = 1$. ($x' = 1$)

3. $f(x) = c, c$ represents a constant. Clearly, $c' = 0$.

4. $f(x) = x^2$. $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} =$

   $\lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} 2x + h = 2x$.

Theorem (differentiation vs. continuity)
If f is differentiable at c then f is continuous at c.
Proof.
If $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = L$ then

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} \lim_{x\to c}(x-c) = L \lim_{x\to c}(x-c) = 0.$$

Hence,

$$0 = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \lim_{x\to c}(x-c) =$$

(we can use the product rule because both limits exist)

$$\lim_{x\to c}(f(x)-f(c)) \frac{x-c}{x-c} =$$

$$\lim_{x\to c}(f(x)-f(c)).$$ QED

Theorem (Arithmetic properties of derivative)

1. $c' = 0$ (derivative of every constant function is zero),
2. $\left(cf(x)\right)' = cf'(x)$,
3. $(f+g)'(x) = f'(x) + g'(x)$,
4. $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$,
5. $\left(\frac{1}{f(x)}\right)' = \frac{-f'(x)}{f^2(x)}$.
6. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

Proof. 1, 2 and 3 are obvious (by properties of limits). 5 follows from 1 and 4 by putting $g(x) = \frac{1}{f(x)}$. 6 follows from 4 and 5.

Proof (of 4, for the curious)

$$[f(x)g(x)]' = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} =$$

$$\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h} =$$

$$\lim_{h\to 0}\frac{f(x+h)(g(x+h)-g(x))+(f(x+h)-f(x))g(x)}{h} =$$

$$\lim_{h\to 0}\left(\frac{f(x+h)(g(x+h)-g(x))}{h} + \frac{(f(x+h)-f(x))g(x)}{h}\right) =$$

$$\lim_{h\to 0}\left(f(x+h)\frac{(g(x+h)-g(x))}{h} + \frac{(f(x+h)-f(x))}{h}g(x)\right) =$$

$$\lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{(g(x+h)-g(x))}{h} + \lim_{h\to 0}\frac{(f(x+h)-f(x))}{h}\lim_{h\to 0} g(x) =$$

$f(x)g'(x) + f'(x)g(x)$. QED

Theorem (Chain rule, derivative of the composition of functions)

$$(f \circ g)'(x) = f'\big(g(x)\big)g'(x)$$

Proof. Skipped, it is beyond the scope of this course.

Corollary. (Derivative of the inverse function)

If the function f^{-1} exists, then

$\big(f^{-1}(x)\big)' = \frac{1}{f'(y)}$, where $f(y) = x$.

Proof.

The chain rule with $g = f^{-1}$ yields:

$(f \circ f^{-1})'(x) = f'\big(f^{-1}(x)\big)(f^{-1})'(x) = 1$, because

$(f \circ f^{-1})'(x) = x' = 1$. Hence, $(f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{f'(y)}$.

QED

Theorem. (Basic formulas)

$(x^n)' = nx^{n-1}$ for $n \in \mathbb{Z} \setminus \{0\}$

$(x^a)' = ax^{a-1}$ for $a \in \mathbb{R}$ and $x > 0$,

$(\sin x)' = \cos x$,

$(\cos x)' = -\sin x$,

$(\tan x)' = \frac{1}{\cos^2 x}$ for $x \neq k\pi + \frac{\pi}{2}$,

$(\cot x)' = -\frac{1}{\sin^2 x}$ for $x \neq k\pi$,

$(e^x)' = e^x$,

$(a^x)' = a^x \ln a$, for $a > 0$ and $a \neq 1$,

Theorem. (Basic formulas cont'd)

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, where $|x| \neq 1,$

$(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$, when $|x| \neq 1,$

$(\arctan x)' = \frac{1}{1+x^2},$

$(ln\ x)' = \frac{1}{x}$, $x > 0,$

$(\log_a x)' = \frac{1}{xln\ a}$ for $x, a > 0$ and $a \neq 1$